AD-776 237

ON BALANCED MATRICES

A. J. Hoffman, et al

IBM Thomas J. Watson Research Center

Prepared for:

Department of the Army
Office of Naval Research
National Science Foundation

14 February 1974

DISTRIBUTED BY:

NTIS

National Technical Information Service
U. S. DEPARTMENT OF COMMERCE
5285 Port Royal Road, Springfield Va. 22151

AD776237

IBM Research

ON BALANCED MATRICES

A. J. Hoffman/D. R. Fulkerson/Rosa Oppenheim

February 14, 1974

RC 4724

Yorktown Heights, New York

San Jose, California

Zurich, Switzerland

2b

ON BALANCED MATRICES

by

A. J. Hoffman†
Mathematical Sciences Department
IBM Thomas J. Watson Research Center
Yorktown Heights, New York 10598

D. R. Fulkerson*
Department of Operations Research
Cornell University

Rosa Oppenheim
Graduate School of Business
Rutgers University

* The work of this author was supported in part by N.S.F. Grant GP-32316X and by O.N.R. Grant N00014-67A-0077-[redacted].
0002

† The work of this author was supported in part by the U.S. Army under contract #DAHC04-72-C-0023.

RC 4724 (#21024)
February 14, 1974
Mathematics

Copies may be requested from:
IBM Thomas J. Watson Research Center
Post Office Box 218
Yorktown Heights, New York 10598

1a

1. INTRODUCTION

In his interesting paper [2], Claude Berge directs our attention to two questions, relevant to the use of linear programming in combinational problems. Let A be a $(0,1)$ - matrix, w and c nonnegative integral vectors, and define the polyhedra

(1.1) $P(A,w,c) = \{y \mid y A \geq w,\ 0 \leq y \leq c\}$,

(1.2) $Q(A,w,c) = \{y \mid yA \leq w,\ 0 \leq y \leq c\}$.

Let $1 = (1,\ldots,1)$ denote the vector all of whose components are 1. The two questions are:

(1.3) If $P(A,w,c)$ is not empty, is the minimum value of $1 \cdot y$, taken over all $y \in P(A,w,c)$, achieved at an integral vector y?

(1.4) Is the maximum value of $1 \cdot y$, taken over all $y \in Q(A,w,c)$, achieved at an integral vector y?

Berge defines a $(0,1)$ - matrix A to be balanced if A contains no square submatrix of odd order whose row and column sums are all two. He shows that the answer to (1.3) is affirmative for all $(0,1)$ - vectors w and c if and only if A is balanced. He shows that the answer to (1.4) is affirmative for all w whose components are 1 or ∞ and for all $(0,1)$-vectors c if and only if A is balanced. Finally, he remarks that for all c whose components are 0 or ∞ and all w whose components are nonnegative integers, the Lovász - Fulkerson perfect graph theorem [4], [6], [7] implies that the answer to (1.3) is affirmative if and only if A is balanced.

In this paper we prove that if A is balanced, then the answers to (1.3) and (1.4) are affirmative for all nonnegative integral w and c.

We do not use the perfect graph theorem as a lemma, nor the results of Berge in [2] or in earlier work on balanced matrices [1].

The above results and those of Berge are used to relate the theory of balanced matrices to those of blocking pairs of matrices and anti-blocking pairs of matrices [3], [4], [5]. We summarize below some pertinent aspects of these two geometric duality theories.

We first discuss briefly the blocking theory. Let A be a nonnegative m by n matrix, and consider the convex polyhedron

$$(1.5) \quad \{x \mid A\,x \geq 1,\ x \geq 0\}.$$

A row vector a^i of matrix A is inessential (does not represent a facet of (1.5)) if and only if a^i is greater than or equal to a convex combination of other rows of A. The (nonnegative) matrix A is proper if none of its rows is inessential. Let A be proper with rows $a^1,\ldots,a^m$. Let B be the r by n matrix having rows $b^1,\ldots,b^r$, where $b^1,\ldots,b^r$ are the extreme points of (1.5). Then B is proper and the extreme points of the polyhedron

$$(1.6) \quad \{x \mid B\,x \geq 1,\ x \geq 0\}$$

are $a^1,\ldots,a^m$. The matrix B is called the blocking matrix of A and vice-versa. Together A and B constitute a blocking pair of matrices, and the polyhedra (1.5) and (1.6) they generate are called a blocking pair of polyhedra. (Thus for any blocking pair of polyhedra, the non-trivial facets of one and the extreme points of the other are represented by exactly the same vectors; trivial facets are those corresponding to the nonnegativity constraints.)

Let A be a nonnegative m by n matrix and consider the packing program

$$(1.7) \quad \text{maximize } 1\cdot y \text{ subject to } yA \leq w,\ y \geq 0,$$

where w is nonnegative. Let B be an r by r nonnegative matrix having rows $b^1,\ldots,b^r$. The max - min equality is said to hold for the ordered pair A,B if, for every n-vector $w \geq 0$, the packing program (1.7) has a solution vector y such that

$$(1.8) \quad 1\cdot y = \min_{1\leq j\leq r} b^j\cdot w\ .$$

One theorem about blocking pairs asserts that the max - min equality holds for the ordered pair of proper matrices A,B if and only if A and B are a blocking pair. Hence, if the max - min equality holds for A,B, it also holds for B,A. (Note that the addition of inessential rows to either A or B does not affect the max - min equality.)

Now let A be a proper (0,1) - matrix, with blocking matrix B. The strong max - min equality is said to hold for A,B if, for any nonnegative integral vector w, the packing program (1.7) has an integral solution vector y, which of course satisfies (1.8). A necessary, but not sufficient, condition for the strong max - min equality to hold for A,B is that each row of B be a $(0,1)$ - vector. To say that an m by n $(0,1)$ - matrix A is proper is simply to say that A is the incidence matrix of m pairwise non-comparable subsets of an n-set, i.e. A is the incidence matrix of a clutter. If the strong max - min equality holds for A and its blocking matrix B, then B is the incidence matrix of the blocking clutter, i.e. B has as its rows all $(0,1)$ - vectors that make inner product at least 1 with all rows of A, and that are minimal with respect to this property. If A and B are a blocking pair of $(0,1)$ - matrices, the strong max - min equality may hold for A,B, but need not hold for B,A. This is in decided contrast with the similar situation for anti-blocking pairs of matrices, which we next briefly discuss.

Let A be an m by n nonnegative matrix with rows $a^1,\ldots,a^m$, having no zero columns, and consider the convex polyhedron

(1.9) $\{x \mid A\,x \leq 1,\ x \geq 0\}$.

(While a row vector a^i of A is inessential in (1.9) if and only if a^i is less than or equal to a convex combination of other rows of A, we shall not limit A to "proper" matrices in this discussion, as we did for blocking pairs, because there will not be a one-one correspondence between non-trivial facets of one member of a pair of anti-blocking polyhedra and the extreme points of the other.) Let D be the r by n matrix having rows $d^1,\ldots,d^r$ where $d^1,\ldots,d^r$ are the extreme points of (1.9). Then D is nonnegative, has no zero columns, and the extreme points of

(1.10) $\{x \mid D\,x \leq 1,\ x \geq 0\}$

are $a^1, \ldots, a^m$ and all projections of $a^1, \ldots, a^m$. D is called an anti-blocking matrix of A, and vice-versa. Together A and D constitute an anti-blocking pair of matrices, and the polyhedra (1.9) and (1.10) are an anti-blocking pair of polyhedra.

Now consider the covering program

(1.11) minimize $1 \cdot y$ subject to $yA \geq w$, $y \geq 0$,

where w is nonnegative. Let D be an r by n nonnegative matrix having no zero columns with rows $d^1, \ldots, d^r$. The min - max equality is said to hold for the ordered pair A,D if, for every n-vector $w \geq 0$, the covering program (1.11) has a solution vector y satisfying

(1.12) $$1 \cdot y = \max_{1 \leq j \leq r} d^j \cdot w.$$

Then the min - max equality holds for A,D if and only if A and D are an anti-blocking pair. Hence, if the min - max equality holds for A,D, it also holds for D,A.

Now let A be a (0,1) - matrix, with anti-blocker D. The strong min - max equality is said to hold for A,D if, for every nonnegative integral vector w, the covering program (1.11) has an integral solution vector y; y of course satisfies (1.12). A necessary and sufficient condition for the strong min - max equality to hold for A,D is that all the essential rows of D be (0,1) - vectors. Hence, if the strong min - max equality holds for A,D, it also holds in the reverse direction D,A (where we may limit D to its essential rows.) In this case it can be shown that the essential (maximal) rows of A are the incidence vectors of the cliques of a graph G on n vertices, and the essential rows of D are the incidence vectors of the anti-cliques (maximal independent sets of vertices) of G. Graph G is thus pluperfect, or equivalently, perfect. The fact that the strong min - max equality for A,D implies the strong min - max equality for D,A is the essential content of the perfect graph theorem.

We shall show in Section 5 that the results described above and those of Berge imply: (a) If A is balanced and B is the blocking matrix of A, then the strong max - min equality holds for both A,B and B,A, and (b) If A is balanced and if D is an anti - blocking matrix of A, then the strong min - max equality holds for A,D (and hence for D,A).

2. VERTICES OF SOME POLYHEDRA.

We first state the lemmas of this section, and then give their proofs.

Lemma 2.1. *If A is balanced, and if $\{x \mid Ax = 1, x \geq 0\}$ is not empty, then every vertex of this polyhedron has all coordinates 0 or 1.*

Lemma 2.2. *If A is balanced, and if $\{x \mid Ax \geq 1, x \geq 0\}$ is not empty, then every vertex of this polyhedron has all coordinates 0 or 1.*

Lemma 2.3. *If A is balanced, and if $\{x,z \mid Ax-z = 1, x \geq 0, z \geq 0\}$ is not empty, then every vertex of this polyhedron is integral.*

Lemma 2.4. *If A is balanced, then every vertex of $\{x,z \mid Ax-z \leq 1, x \geq 0, z \geq 0\}$ is integral. Hence if A is balanced, every vertex of $\{x \mid Ax \leq 1, x \geq 0\}$ has coordinates 0 or 1.*

Note that Lemma 2.1 is a special case of Lemma 2.3, but it is convenient to separate the proofs.

Proof of Lemma 2.1. If A is balanced, then every submatrix of A is balanced. We shall prove Lemma 2.1 by induction on the number of rows of A. It is clearly equivalent to prove that if $x > 0$ satisfies $Ax = 1$, then there exists a set of non-overlapping columns $a_{j_1}, \ldots, a_{j_k}$ of A

(i.e., $a_{j_r} \cdot a_{j_s} = 0$ for $r \neq s$) whose sum is the vector 1. For any set S of non-overlapping columns, define $C(S)$, the "cover of S", to be the number of i such that $\sum_{j \in S} a_{ij} = 1$. Let S^* be a set of non-overlapping columns such that $C(S^*) \geq C(S)$ for any set S of non-overlapping columns. If $C(S^*) = m$ = number of rows of A, we are done, so assume $C(S^*) = k < m$, and, say, $\sum_{j \in S^*} a_{ij} = 1$ for $i = 1,\ldots,k$. Let $\bar{A}$ be the submatrix of A formed by rows $1,\ldots,k$. We have $\bar{A}x = 1$, $x > 0$, so, by the induction hypothesis, any column of $\bar{A}$ is contained in a set T of non-overlapping columns of $\bar{A}$ such that $C(T) = k$. In particular, let j^* be a column index such that $a_{ij^*} = 1$ for some $i \in \{k+1,\ldots,m\}$, and let the aforementioned T contain j^*. Now some column indices in T (possibly none) may coincide with some column indices in S^*. Let $V = T - S^*$, $U = S^* - T$, both non-empty. Define a graph $G(\bar{A})$ whose points are the indices in $V \cup U$, with j and ℓ adjacent if and only if $\bar{a}_j \cdot \bar{a}_\ell > 0$. Clearly $G(\bar{A})$ is bipartite with parts U and V. Let W be the vertices of the connected component $W(\bar{A})$ of $G(\bar{A})$ containing j^* (W may be $V \cup U$). It follows that

(2.1) for $i = 1,\ldots,k$, $\sum_{j \in U \cap W} a_{ij} = \sum_{j \in V \cap W} a_{ij} = 0$ or 1.

Suppose that, for each $i = k + 1,\ldots,m$,

(2.2) $\sum_{j \in V \cap W} a_{ij} \leq 1$.

Since $j^* \in W$, it follows from (2.1) and (2.2) that the columns of A with indices in

$$(S^* - (U \cap W)) \cup (V \cap W)$$

are a non-overlapping set of columns with cover $\geq k + 1$, contradicting the definition of S^*. Hence, (2.2) is untenable. Now consider the graph $W(A)$ with point set W, where j and ℓ are adjacent if and only if $a_j \cdot a_\ell > 0$. Recall that $W(\bar{A})$ is connected and bipartite. The graphs $W(\bar{A})$ and $W(A)$ have the same point set W, but $W(A)$ has more edges. In particular, there exists at least one pair of points in $W \cap V$ which are adjacent in $W(A)$. Let j and ℓ be points in $W \cap V$ such that the shortest path P in $W(\bar{A})$ joining j and ℓ contains no points j' and ℓ' in $W \cap V$ adjacent in $W(A)$ other than j and ℓ. Clearly such a path exists and is of even length. Let this path be

$$j = j_1,\ i_1,\ j_2,\ i_2, \ldots, j_p,\ i_p,\ j_{p+1} = \ell$$

where the first, third, fifth,... indices are in V, the second, fourth,... indices are in U. Let $r^* \in \{k + 1, \ldots, m\}$ satisfy $a_{r^* j_1} = a_{r^* j_{p+1}} = 1$ and choose $r_1, \ldots, r_p$, $s_1, \ldots, s_p$ such that

$$a_{r_t j_t} = a_{r_t i_t} = 1, \quad t = 1, \ldots, p,$$

$$a_{s_t i_t} = a_{s_t j_{t+1}} = 1, \quad t = 1, \ldots, p.$$

That such indices exist follows from the construction of the path P. It is now clear that the submatrix of A formed by the columns $i_1, \ldots, i_p, j_1, \ldots, j_{p+1}$ and rows $r^*, r_1, \ldots, r_p, s_1, \ldots, s_p$ violates the hypothesis that A is balanced. Thus $C(S^*) = m$, proving Lemma 2.1.

<u>Proof of Lemma 2.2.</u> If x is a vertex of $\{x \mid A\,x \geq 1,\ x \geq 0\}$, it is a vertex of the polyhedron obtained by deleting the inequalities of $Ax \geq 1$ that are strict. By Lemma 2.1, every vertex of this polyhedron has all coordinates 0 or 1.

Proof of Lemma 2.3. If (x,z) is a vertex of $\{x,z \mid A\,x-z = 1, x \geq 0, z \geq 0\}$, then x is a vertex of $\{x \mid A\,x \geq 1,\ x \geq 0\}$. Lemma 2.3 thus follows from Lemma 2.2.

Proof of Lemma 2.4. If (x,z) is a vertex of $\{x,z \mid A\,x-z \leq 1, x \geq 0,\ z \geq 0\}$, it is a vertex of the polyhedron obtained by deleting the inequalities of $Ax-z \leq 1$ that are strict. Thus Lemma 2.4 follows from Lemma 2.3.

3. Solution of Problem (1.3). We first prove a lemma.

Lemma 3.1. Let A be a $(0,1)$ - matrix satisfying the condition: For all nonnegative integral vectors w and c such that $P(A,w,c)$ is not empty, the minimum value of $1\cdot y$, $y \in P(A,w,c)$, is an integer. Then for all nonnegative integral vectors w and c such that $P(A,w,c)$ is not empty, there exists an integral vector y that minimizes $1\cdot y$ over $y \in P(A,w,c)$.

Proof. The lemma is true if $1\cdot c = 0$, and so we argue by induction on $1\cdot c$.

Assume $y = (y_1, y_2, \ldots, y_m)$ is a solution to the linear program

(3.1) minimize $1\cdot y$ subject to $y \in P(A,w,c)$,

with at least one component not integral, say $y_1 = r + \theta$, where $r \geq 0$ is an integer and $0 < \theta < 1$. Let $1\cdot y = k$, where k is an integer. For any number z, define $z^+ = \max(0,z)$, and for any vector $z = (z_1, z_2, \ldots)$, define $z^+ = (z_1^+, z_2^+, \ldots)$. Let $\alpha = (r, y_2, \ldots, y_m)$, and note that $0 \leq \alpha \leq \tilde{c} = (c_1-1, c_2, \ldots, c_m)$. Let a^1 be the first row of A. Since

$\alpha A \geq w - a^1$ and $\alpha A \geq 0$, we have $\alpha A \geq (w - a^1)^+$. Thus $\alpha \in P(A, (w-a^1)^+, \tilde{c})$, and $1 \cdot \alpha = k - \theta < k$. Now $1 \cdot \tilde{c} < 1 \cdot c$. Hence, by the induction assumption there exists an integral vector $\beta = (\beta_1, \ldots, \beta_m)$ such that $\beta A \geq (w - a^1)^+ \geq w - a^1$, $0 \leq \beta \leq \tilde{c}$, and $1 \cdot \beta = \ell \leq k - \theta < k$, where ℓ is an integer. Therefore, the integral vector $\bar{\beta} = (\beta_1 + 1, \beta_2, \ldots, \beta_m) \in P(A,w,c)$, $1 \cdot \bar{\beta} = \ell + 1 \leq k$. But no solution to (3.1) can have value less than k, and hence $1 \cdot \bar{\beta} = k$. Thus $\bar{\beta}$ is an integral vector solving (3.1).

Theorem 3.2. *Let A be balanced, and let w and c be nonnegative integral vectors such that $P(A,w,c)$ is not empty. Then the linear program (3.1) has an integral solution.*

Proof. Since $P(A,w,c)$ is not empty and bounded, (3.1) has a solution. Hence, by the duality theorem of linear programming, the dual program

(3.2) maximize $w \cdot x - c \cdot z$ subject to $Ax - z \leq 1$, $x \geq 0$, $z \geq 0$,

has a solution. One such must occur at a vector with integral coordinates, by Lemma 2.4, so the common value of (3.2) and of (3.1) is an integer. But this means that the hypothesis of Lemma 3.1 holds. Hence, the conclusion of Lemma 3.1 holds, proving the theorem.

Note that the theorem holds if all coordinates of the vector c are ∞, an observation we will need below.

4. *Solution of Problem* (1.4). We denote this section to the proof of Theorem 4.1 below.

Theorem 4.1. Let A be a balanced matrix, and let w and c be nonnegative integral vectors. Then the linear program

(4.1) *maxmize* $1 \cdot y$ *subject to* $y \in Q(A,w,c)$

has an integral solution vector y.

Proof. We first remark that if A is balanced, the matrix (A,I) is balanced. Thus it suffices to prove that if A is balanced and $w \geq 0$ is integral, then the linear program

(4.2) maximize $1 \cdot y$ subject to $yA \leq w$, $y \geq 0$,

has an integral solution vector y. We shall prove this by a double induction on the pair of integers $(1 \cdot w, m)$, where A has m rows. Note that the theorem clearly is valid for any $m \geq 1$ if $1 \cdot w = 0$; it is also valid for any nonnegative integer value of $1 \cdot w$ if $m = 1$ (i.e., if (4.2) is a problem in one variable.)

Let $y = (y_1, y_2, \ldots, y_m)$ be a fractional solution of (4.2). If at least one y_i is zero, we are in the situation described by the pair of integers $(1 \cdot w, m-1)$, since any submatrix of A is balanced, and the induction hypothesis applies. Thus we suppose all $y_i > 0$. By Lemma 2.2 and the duality theorem of linear programming, we know that $1 \cdot y = k$, where k is an integer. Now suppose there is at least one j such that $y \cdot a_j < w_j$, where a_j is the jth column of A. Thus $w_j > 0$. If $y \cdot a_j \leq w_j - 1$, we consider the pair of integers $(1 \cdot w - 1, m)$. By the inductive hypothesis, there is an integral vector z such that $zA \leq 0$, $z \geq 0$, $1 \cdot z = 1 \cdot y = k$, and we are done. Thus we may assume that $y \cdot a_j = w_j - 1 + \theta$, where $0 < \theta < 1$.

Hence $a_j \neq 0$. Then clearly we can find a vector z such that $z \geq 0$, $zA \leq (w_1, w_2, \ldots, w_j - 1, \ldots, w_n)$, $z \leq y$, and $1 \cdot z = k - \theta$. By the inductive hypothesis for the pair of integers $(1 \cdot w - 1, m)$, there is an integral vector α satisfying $\alpha \geq 0$, $\alpha A \leq (w_1, \ldots, w_j - 1, \ldots, w_n) \leq w$, $1 \cdot \alpha \geq k - \theta$, hence $1 \cdot \alpha = k$, and we are done.

Thus $y \cdot a_j = w_j$ for all j and $y_i > 0$ for all i. By the principle of complementary slackness, every optimal solution of the dual problem

(4.3) minimize $w \cdot x$ subject to $Ax \geq 1$, $x \geq 0$

satisfies $Ax = 1$, $x \geq 0$, $w \cdot x = k$. Select one such x. Then y and x are optimal solutions, respectively, of the dual programs

(4.4) minimize $1 \cdot y$ subject to $yA \geq w$, $y \geq 0$,

(4.5) maximize $w \cdot x$ subject to $Ax \leq 1$, $x \geq 0$,

with common value $1 \cdot y = w \cdot x = k$. By the remark at the end of the last section, there exists an integral vector α such that $\alpha \geq 0$, $\alpha A \geq w$, $1 \cdot \alpha = k$. If $\alpha A = w$, we are done. So assume $\alpha \cdot a_j > w_j$ for at least one j. Since $y_i > 0$ for all i, there is a number t, $0 < t < 1$, such that $y_i > (1-t)\alpha_i$ for all i. Let vector z solve $y = (1-t)\alpha + tz$, i.e., $z = \frac{1}{t}[y-(1-t)\alpha]$. Thus $z \geq 0$ and $1 \cdot z = k$. Now, since $yA = w$ and $\alpha A \geq w$, it follows that $zA \leq w$. Moreover, since there is a j such that $\alpha \cdot a_j > w_j$, we have $z \cdot a_j < w_j$. Thus z is a solution to (4.1) with $z \cdot a_j < w_j$ for some j. However, as we have already seen, in this case the theorem is true by induction, and this completes the proof of Theorem 4.1.

5. Blocking pairs and anti-blocking pairs. Our purpose in this section is to prove the following theorems, which were mentioned in Section 1.

Theorem 5.1. *Let* A *be balanced and let* B *be the blocking matrix of* A. *Then the strong max - min equality holds for both* A,B *and* B,A.

Theorem 5.2. *Let* A *be balanced with no zero columns and let* D *be an anti-blocking matrix of* A. *Then the strong min - max equality holds for both* A,D *and* D,A.

Note that we have not assumed the (0.1) - matrix A in the statement of Theorem 5.1 to be proper; it would be no restriction to do so, however; we could just consider the minimal (essential, in the blocking sense) rows of A.

Proof of Theorem 5.1. That the strong max - min equality holds for the ordered pair A,B follows from Theorem 4.1 by taking the components of the vector c in Theorem 4.1 all equal to ∞.

To show that the strong max - min equality holds in the reverse direction B,A, we first note that Theorem 2 of [2] can be rephrased in blocking terminology as follows: Let A be balanced and let B have as its rows all (0,1) - vectors that make inner product at least 1 with every row of A and that are minimal with respect to this property (i.e., B is the incidence matrix of the blocking clutter of the clutter of minimal rows of A); then the linear program

(5.1) maximize $1 \cdot y$ subject to $yB \leq 1$, $y \geq 0$, has a (0,1) solution vector y satisfying $1\cdot y = \min 1\cdot a^i$, taken over all rows a^i of A. To get the strong max-min equality for B,A from this, we need to pass from the vector 1 on the right-hand side of $yB \leq 1$ to a general nonnegative integral vector w. This transformation can be effected inductively by first observing that if A is balanced, and if we duplicate a column of A, the resulting matrix A' is balanced. (Prop. 5 of [2]).

Pictorially:

$$\begin{array}{ccc}
A \text{ (balanced)} & & A' \text{ (balanced)} \\
\left[\begin{array}{c|c}
1 & \\
\vdots & \\
1 & X \\
0 & \\
\vdots & \\
0 &
\end{array}\right]
& \rightarrow &
\left[\begin{array}{cc|c}
1 & 1 & \\
\vdots & \vdots & \\
1 & 1 & X \\
0 & 0 & \\
\vdots & \vdots & \\
0 & 0 &
\end{array}\right]
\end{array}$$

$$\begin{array}{ccc}
B \text{ (blocker of } A) & & B' \text{ (blocker of } A') \\
\left[\begin{array}{c|c}
1 & \\
\vdots & Y \\
1 & \\ \hline
0 & \\
\vdots & Z \\
0 &
\end{array}\right]
& \rightarrow &
\left[\begin{array}{cc|c}
1 & 0 & \\
\vdots & \vdots & Y \\
1 & 0 & \\ \hline
0 & 1 & \\
\vdots & \vdots & Y \\
0 & 1 & \\ \hline
0 & 0 & \\
\vdots & \vdots & Z \\
0 & 0 &
\end{array}\right]
\end{array}$$

Thus, if the first component of w is 2, instead of 1, we can consider the linear program

(5.2) maximize $1\cdot y$ subject to $yB' \leq 1$, $y \geq 0$, instead of

(5.3) maximize $1 \cdot y$ subject to $yB \leq (2,1,\ldots,1)$, $y \geq 0$.

It follows that a general nonnegative integral vector w can be dealt with by deleting certain columns of A (those corresponding to zero components of w), replicating others, yielding a new balanced matrix, and making the appropriate transformations on the blocker B of A (a zero component of w means that we delete the corresponding column of B and also delete all rows of B that had a 1 in that column). In this way, one can deduce from Theorem 2 of [2] that if A is balanced, the strong max - min equality holds for B,A.

In connection with Theorem 5.1 and its proof, we point out that the blocking matrix B of a balanced matrix A may not be balanced. For example, let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Matrix A is balanced, with blocking matrix

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

<u>Proof of Theorem 5.2.</u> If the $(0,1)$ - matrix A has no zero columns, then $P(A,w,c)$ is not empty, where c is the vector all of whose components are ∞. The strong min - max equality for A,D, where D is an anti-blocking

matrix of A, now follows from Theorem 3.2 and the discussion in Section 1 concerning anti-blocking pairs. Moreover, as noted in Section 1, the strong min - max equality for A,D implies the strong min - max equality for D,A.

Theorem 5.2 can be paraphrased as follows. The maximal (essential, in the anti-blocking sense) rows of a balanced matrix A are the incidence vectors of the cliques of a perfect graph G. Consequently the essential rows of D are the incidence vectors of the anti-cliques of G.

REFERENCES

1. Claude Berge, Graphes et Hypergraphes, Dunod, Paris, 1970 (ch. 20).

2. ____________, Balanced matrices, Math. Prog. 2 (1972) 19-31.

3. D. R. Fulkerson, Blocking polyhedra, in Graph Theory and its Applications, Academic Press, New York, 1970, 93-112.

4. ______________, Anti-blocking polyhedra, Jour. Comb. Th. 12, No. 1 (1972) 50-71.

5. ______________, Blocking and anti-blocking pairs of polyhedra, Math. Prog. 1 (1971) 168-194.

6. ______________, On the perfect graph theorem, in Mathematical Programming, Academic Press, New York, 1973, 69-76.

7. L. Lovász, Normal hypergraphs and the perfect graph conjecture, Discrete Math. 2 (1972) 253-267.